The Difference Of All Sequences In Weighted Composition Operators Acting From $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$ spaces

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الملخص

تتناول هذه الدراسة محدودية واتساق اختلافات عوامل التركيب الموزونة من فضاء بلوخ الفا إلى فضاء بلوخ بيتا على قرص الوحدة المفتوح. وترتبط هذه الدراسة بالبناء االتوبولوجي للتركيب الموزون من فضاء بلوخ الفا إلى فضاء بلوخ بيتا.

Abstract

This study the restricted and consistency of the differences of weighted composition operators from $\alpha-$ Bloch space to $\beta-$ Bloch space on open unit disk. This study has a connection to the topological building of weighted composition from $\alpha-$ Bloch space to $\beta-$ Bloch space .

Keywords: differences, weighted composition operators, $\alpha-$ Bloch space , $\beta-$ Bloch space

1.Introduction.

Take $\mathbb{R}$ be an open disk in the complex plane $\mathbb{O}$ and $H(\mathbb{R})$ be the class of all functions analytic on $\mathbb{R}$. The researcher denotes by $S(\mathbb{R})$ the set of all analytic self-maps on $\mathbb{R}$. for (1+ϵ) $\in H(\mathbb{R})$, denote by $M_{(1+\epsilon)}$ the multiplication operator. For $\varphi_r \in S(\mathbb{R})$, denote by C_{φ_r} the composition operators. NiNG XU and ZE-HUA ZHOU* [17] . The researcher intends to make few specific changes. Then the weighted composition operator $(1+\epsilon)C_{\varphi_r}$ is a linear transformation of $H(\mathbb{R})$ denote by

$$(1+\epsilon)C_{\varphi_r}f^2(z) = \left(M_{1+\epsilon}\, C_{\varphi_r}\right)f^2(z) = (1+\epsilon)(z)f^2(\varphi_r)(z)).$$

For $-1 < \epsilon < \infty$, recall that the Bloch type space $\mathcal{B}^\alpha$, or $\alpha -$ Bloch space, consists of all $f^2 \in H(\mathbb{R})$ such that

$$\|f^2\|_\alpha = \sup_{z \in \mathbb{R}} (1-|z|^2)^\alpha \left|f^{2'}\right| < \infty.$$

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm $\|f^2\|_{\mathcal{B}^\alpha} - \|f^2\|_\alpha = |f^2(0)|$.

The little $\alpha -$ Bloch $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}^\alpha$, consisting of all $f^2 \in H(\mathbb{R})$ such that

$\lim_{|z| \to 1} (1-|z|^2)^{1-\epsilon} \left|f^{2'}(z)\right| = 0.$ When $\epsilon = 0$, $\mathcal{B}^\alpha$ is classical space $\mathcal{B}$.

For z ,($z+\epsilon$) $\in \mathbb{R}$, the pseudo-hyperbolic distance $\rho(z, z+\epsilon)$ between z and $z+\epsilon$ is given by

$$\rho(z, z+\epsilon) = \left|\frac{\epsilon}{1-\overline{(z+\epsilon)}z}\right|.$$

For $(1+\epsilon\ ) \in \mathbb{R}$, Let $\sigma_{(1+\epsilon)}$ be the Möbius transformation of $\mathbb{R}$ defines by

$$\sigma_{(1+\epsilon)}(z) = \frac{(1+\epsilon)-z}{1-\overline{(1+\epsilon)}z}.$$

The researcher remarks that $\rho(1+\epsilon, z) = \left|\sigma_{(1+\epsilon)}(z)\right| \le 1$. For $\varphi_r \in S(\mathbb{R})$, the Schwarz-Pick type derivative of φ_r is defined by

$$\varphi_r(z) = \frac{(1-|z|^2)^{1+\epsilon}}{(1-|\varphi_r z|^2)^{1+\epsilon}}\ \varphi_r'(z)$$

The Schwarz-Pick Lemma implies that $\left\|\varphi_r^\#\right\|_\infty - 1 \le 0$. Madigan and Matheson [10] proved that C_{φ_r} is compact on $\mathcal{B}$ if and only if $|\varphi_r(z)| \longrightarrow 0$ wherever $|\varphi_r(z)| \longrightarrow 1$. Ohno and Zhao [13] generalized this result [16] to the case of $(1+\epsilon)C_{\varphi_r}$ on $\mathcal{B}$.

This study of the difference of two composition operators was started on the Hardy space H^2. The main purpose for this study is to understand the topological structure of $\mathcal{C}(H^2)$, the set of composition operators of $\mathcal{C}(H^2)$, see [15]. This work gave a relationships between a component problem and the behavior of the difference of two composition operators acting from $\mathcal{B}\ to\ H^\infty$. After that , such related problems have been studied on several spaces of analytic functions by many author, see [1,7,4,9,14] for the study of differences of composition operators on various spaces. Some related results concerning the differences of weighted composition operators on various spaces can be founded in, for example,[2,5,6,8].

As an analogue, the topological structure of the set $C(\mathcal{B}^{\alpha} \longrightarrow \mathcal{B}^{\beta})$ of bounded weighted composition operators from $\mathcal{B}^{\alpha}\ to\ \mathcal{B}^{\beta}$ can be considered. In this context, we deal with the differences of weighted composition operators from $\mathcal{B}^{\alpha}\ to\ \mathcal{B}^{\beta}$ c when $\epsilon > 0$. The main purpose of this paper is to express the boundedness and compactness of $uC_{\varphi_r} - vC_{\psi_r}$ from $\mathcal{B}^{\alpha}\ to\ \mathcal{B}^{\beta}$ which generalizes [6]. The authors expect that these results will play some roles in the succeeding investigation.

For two quantities A and B which may depend on $\varphi_r\ and\ \psi_r$, we use the abbreviation $A \leq B$
whenever this is a positive constant C(independent of $\varphi_r\ and\ \psi_r$) such that $A \leq CB$. We write $A \sim B$ if $A \lesssim B \lesssim A$.

2- Basic requirements

The researcher collects some basic properties of functions in $\mathcal{B}^{\alpha}$. It is known that the follows hold (see [16]), for $(z\, , z+\epsilon\ ) \in \mathbb{R}$

$$d_{\mathbf{1+\epsilon}}(z\, , z+\epsilon) = sup\{|f^2(z) - f^2(z+\epsilon)|: \|f^2\|_{\mathcal{B}^{\alpha}} \leq 1\},$$

$$|f^2(z)| \leq \frac{\|f^2\|_{\mathcal{B}^{1+\epsilon}}}{(1-|z|^2)^{\epsilon}}\ , f^2 \in \mathcal{B}^{1+\epsilon}\, , \epsilon > 0.$$

The researcher uses the following notion:

$$(1+\epsilon)(z) = (1+\epsilon)\big(\varphi_r(z), \psi_r(z)\big), \qquad \sigma(z) = \frac{\varphi_r - \psi_r(z)}{\left(1-\overline{\varphi_r}\psi_r(z)\right)^2}\ ,$$

$$\tau(z) = \frac{(1-|\varphi_r(z)|^2)(1-|\psi_r|^2)}{\left(1-\overline{\varphi_r}\psi_r(z)\right)^2}.$$

The researcher remarks that

$$|\sigma\varphi_r(z+\epsilon)\ \sigma\ (z+\epsilon))| = (1+\epsilon)(z+\epsilon)\ , \qquad and\ \ |\tau(z)| = 1-(1+2\epsilon+\epsilon^2)(z).$$

For $\{z_{1-\epsilon}\} \subset \mathbb{R}$, denote that $\varphi_{r_{1-\epsilon}} = \varphi_r(z_{1-\epsilon})$ and $\psi_{r_{1-\epsilon}} = \psi_r(z_{1-\epsilon})$. The researcher uses notion $\sigma_{1-\epsilon}\, , \tau_{1-\epsilon}$.

Note that

$$U(z) = (1-|z|^2)^{1+\epsilon}\ \acute{u}(z),\ \ V(z) = (1-|z|^2)^{1+\epsilon}\ \acute{v}(z)$$

Lemma1. [14] Let $-1 < \epsilon < \infty$. For all $z\, , (z+\epsilon) \in \mathbb{R},$ the Bloch-type induced distance is given by

$$b_{1+\epsilon}(z, z+\epsilon) = \sup_{\|f^2\|_{\mathcal{B}^{1+\epsilon}} \leq 1} \left|(1-|z|^2)^{1+\epsilon} f^{2'}(z) - (1-|z+\epsilon|^2)^{1+\epsilon} f^{2'}(z+\epsilon)\right|.$$

Then

$$b_{1+\epsilon}(z, z+\epsilon) \lesssim \rho(z, z+\epsilon)$$

Remark 1. Particularly, if the researcher takes the function $f^2_{\ltimes}(z) = \frac{(1-|\ltimes|^2)^{\epsilon}}{(1-\bar{\ltimes}z)^{2\epsilon}}$, $h^2_{\ltimes}(z) = \int_0^z f^2_{\ltimes}(\zeta)d(\zeta)$, then $\|h^2_{\ltimes}\|_{\mathcal{B}^{1+\epsilon}} \leq 1$. From Lemma 1, take $\ltimes = \varphi_r(z+\epsilon)$, the researcher has that

$$|1-\tau^{\epsilon}(z+\epsilon)| = \left|1 - \frac{(1-|\varphi_r(z+\epsilon)|^2)^{2\epsilon}}{(1-\overline{\varphi_r}(z+\epsilon)\psi_r(z+\epsilon))^{2\epsilon}}\right|$$

$$= \left|(1-|\varphi_r(z+\epsilon)|^2)^{\epsilon}\, h^2_{\varphi_{r_{(z+\epsilon)}}}(\varphi_r(z+\epsilon)) - (1-|\varphi_r(z+\epsilon)|^2)^{\epsilon}\, h^2_{\varphi_{r_{(z+\epsilon)}}}(\varphi_r(z+\epsilon))\right|$$

$$\leq b_{1+\epsilon}(\varphi_r(z+\epsilon), \psi_r(z+\epsilon) \lesssim (1+\epsilon)(z+\epsilon).$$

The result is right for $|1-\tau^{\epsilon}(z+\epsilon)|$ too.

Lemma 2. There exists a constant $C > 0$ such that for any $a, b \in \mathbb{R}$,

$$\left|\frac{1-|1+\epsilon|^2}{1-\overline{1+\epsilon}\, b}\right|^{\epsilon} \leq C.$$

3- **The boundary From $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$ spaces**

the researcher formulates and prove the main results [17].

Proposition 1. Let $\varphi_r, \psi_r \in S(\mathbb{R})$ and $(1+\epsilon), (1+\epsilon) \in H(\mathbb{R})$. Then $(1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}$ is bounded from $\mathcal{B}^{1+\epsilon}(\epsilon > 0)$ to $\mathcal{B}^{1+\epsilon}$, in the following conditions hold:

- **(i)** $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z)\varphi_r(z) - (1+\epsilon)(z)\psi_r(z)| < \infty.$
- **(ii)** $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z)\varphi_r(z)| + |(1+\epsilon)(z)\, \psi_r\,(z)| < \infty.$
- **(iii)** $\sup_{z \in \mathbb{R}} L(z) < \infty,$

  where

  $L(z) = \min\left\{\frac{U(Z)-V(Z)}{(1-|\varphi_r(z)|^2)^{\epsilon}}, \frac{U(Z)-V(Z)}{(1-|\psi_r(z)|^2)^{\epsilon}}, |U(z)|d_{1+\epsilon}(z)|V(z)|d_{1+\epsilon}(z)\right\}.$

Proof. For any $f^2 \in \mathcal{B}^{1+\epsilon}$ with $\|f^2\|_{\mathcal{B}^{1+\epsilon}} - 1 \leq 0$, the researcher has that

$$\left\|\big((1+\epsilon)\varphi_r(z) - (1+\epsilon)\psi_r\big)(f^2))\ \right\|_{\mathcal{B}^{1+\epsilon}}$$

$$= \left|(1+\epsilon)(0)f^2\big(\varphi_r(0)\big) - (1+\epsilon)(0)f^2(\psi_r)(0))\right|$$

$$+ \left\|((1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}(f^2))\right\|_{\beta}$$

$$\leq |(1+\epsilon)(0) - (1+\epsilon)(0)|f^2(\varphi_r(0)) +$$

$$(1+\epsilon)(0)\left\|f^2\big(\varphi_r(0)\big) - f^2\big(\psi_r(0)\big)\right\| + \left\|((1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}(f^2))\right\|_{\beta}$$

and

$$\left\|((1+\epsilon)C_{\varphi_r}-(1+\epsilon)C_{\psi_r}(f^2))\right\|_\beta$$

$$\le \sup_{z\in\mathbb{R}}\left|(1+\epsilon)(z)\frac{(1-|z|^2)^{1+\epsilon}}{(1-|\varphi_r(z)|^2)^{1+\epsilon}}\varphi_r'(z)\left(1-|\psi_r(z)|^2)^{1+\epsilon}\, f^{2'}(\psi_r)\right)\right.$$

$$\left.-(1+\epsilon)(z)\frac{(1-|z|^2)^{1+\epsilon}}{(1-|\varphi_r(z)|^2)^{1+\epsilon}}\varphi_r'(z)\left(1-|\psi_r(z)|^2)^{1+\epsilon}\, f^{2'}(\psi_r)\right)\right|$$

$$+\sup_{z\in\mathbb{R}}|(1-|z|^2)^{1+\epsilon}\,(1+\epsilon)'(z)f^2(\varphi_r(z))V$$

$$-(1-|z|^2)^{1+\epsilon}\,(1+\epsilon)'(z)f^2(\psi_r(z))|$$

$$\triangleq I+II.$$

First the researcher estimates *I*,

$$I\le \sup_{z\in\mathbb{R}}|(1+\epsilon)(z)\varphi_r(z)-(1+\epsilon)(z)\psi_r(z)|\|f^2\|_{\mathcal{B}^{1+\epsilon}}+\sup_{z\in\mathbb{R}}|(1+\epsilon)(z)\psi_r(z)|b_{1+\epsilon},\varphi_r(z)\psi_r(z))$$

$$\le \sup_{z\in\mathbb{R}}|(1+\epsilon)(z)\varphi_r(z)-(1+\epsilon)(z)\psi_r(z)|+\boldsymbol{C}\sup_{z\in\mathbb{R}}|(1+\epsilon)\psi_r(z)|(1+\epsilon)(z).$$

Similarly $I\le \sup_{z\in\mathbb{R}}|(1+\epsilon)(z)\varphi_r(z)-(1+\epsilon)(z)\psi_r(z)|+\boldsymbol{C}\sup_{z\in\mathbb{R}}|(1+\epsilon)\psi_r(z)|(1+\epsilon)(z)$

Then the researcher estimates *11*.

$$II\le \sup_{z\in\mathbb{R}}\left[\left|\left(U(z)-V(z)\right)f^2\left(\varphi_r(z)\right)\right|+\left|V(z))f^2\left(\varphi_r(z)\right)-f^2(\psi_r(z)))\right|\right]$$

$$\le \sup_{z\in\mathbb{R}}\left[\frac{U(Z)-V(Z)}{(1-|\varphi_r(z)|^2)^{\epsilon}}+V(z)d(1+\epsilon)(z)\right].$$

Similarly $II\le \sup_{z\in\mathbb{R}}\left[\frac{U(Z)-V(Z)}{(1-|\psi_r(z)|^2)^{\epsilon}}+U(z)d(1+\epsilon)(z)\right].$

Note

$$L(z)=\min\left\{\frac{U(Z)-V(Z)}{(1-|\varphi_r(z)|^2)^{\epsilon}},\frac{U(Z)-V(Z)}{(1-|\psi_r(z)|^2)^{\epsilon}},|U(z)|d_{1+\epsilon}(z)|V(z)|d_{1+\epsilon}(z)\right\}.$$

Consider *I and II* , then the proposition holds.

Proposition 2. Let $\varphi_r, \psi_r \in S(\mathbb{R})$ and $(1+\epsilon), (1+\epsilon) \in H(\mathbb{R})$. Then $(1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}$ is bounded from $\mathcal{B}^{1+\epsilon}(\epsilon > 0)$ to $\mathcal{B}^{1+\epsilon}$, in the following conditions hold:

(i) (a) $\sup_{z \in \mathbb{R}} \left|(1+\epsilon)(z)\varphi_{r(z)}\right|(1+\epsilon)^2(z) < \infty$, (b) $\sup_{z \in \mathbb{R}} \left|(1+\epsilon)(z)\psi_{r(z)}\right|(1+\epsilon)^2(z) < \infty$.

(ii) (a) $\sup_{z \in \mathbb{R}} |U((z)|(1+\epsilon)^3(z) < \infty$, $(b) \sup_{z \in \mathbb{R}} |V((z)|(1+\epsilon)^3(z) < \infty$.

(iii) $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z)\varphi_r(z) + (1+\epsilon)(z)\psi_r(z)|(1+\epsilon)(z) < \infty$.

Definition 1. Let $\{z_n\}$ be a sequence in $\mathbb{R}$.

(i) Let $D_{(1+\epsilon),\varphi_r}$ be the set of all sequences $\{z_n\}$ such that $|(1+\epsilon)(z_n)\varphi_r(z_n)| \to \infty$ when $|z_n| \to \infty$

(ii) Let $E_{(1+\epsilon),\varphi_r}$ be the set of all sequences $\{z_n\}$ such that

$$\frac{|U(z_n)|}{(1-|\varphi_r(z_n)|^2)^{\epsilon}} \longrightarrow \infty ,$$

Remark 2. Using these notions, [12] is rewritten that a weighted composition operator (1+ϵ)C_{φ_r}
is bounded from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$ if and only if $D_{(1+\epsilon),\varphi_r} = E_{(1+\epsilon),\varphi_r} = \theta$. the researcher wills characterize the boundedness of $(1+\epsilon)\varphi_r(z) - (1+\epsilon)\psi_r$ from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$ under the assumption

$$\sup_{z \in \mathbb{R}} \frac{|U(z_n)|_{(1+\epsilon)(z)}}{(1-|\varphi_r(z_n)|^2)^{\epsilon}} < \infty \quad \text{and} \quad \sup_{z \in \mathbb{R}} \frac{|V(z_n)|_{(1+\epsilon)(z)}}{(1-|\psi_r(z_n)|^2)^{\epsilon}} < \infty. \qquad \text{(A)}$$

Theorem 1. Let $\varphi_r, \psi_r \in S(\mathbb{R})$ and $(1+\epsilon), (1+\epsilon) \in H(\mathbb{R})$. Suppose that the condition (A) holds. Then $(1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}$ is bounded from $\mathcal{B}^{\alpha}(\epsilon > 0)$ to $\mathcal{B}^{\beta}$, in the following conditions hold:

(i) $D_{(1+\epsilon),\varphi_r} = D_{(1+\epsilon),\psi_r}$

(ii) $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon) - (1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)| < \infty$.

(iii) (a) $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)|(1+\epsilon)(z) < \infty$ (b) $\sup_{z \in \mathbb{R}} |(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|(1+\epsilon)(z) < \infty$.

(iv) $E_{(1+\epsilon),\varphi_r} = E_{(1+\epsilon),\psi_r}$.

(v) (a) $\sup_{z \in E_{(1+\epsilon),\varphi_r}} \lim_{n\to\infty} \frac{|U(Z_n)-V(z_n)|}{(1-|\varphi_r(z_n)|^2)^\epsilon} < \infty$. (b) $\sup_{z \in E_{(1+\epsilon),\psi_r}} \lim_{n\to\infty} \frac{|U(Z_n)-V(z_n)|}{\left(1-\left|\psi_r(z_n)\right|^2\right)^\epsilon} < \infty$.

(vi) (a) $\sup_{z \in E_{(1+\epsilon),\varphi_r}} \lim_{n\to\infty} |U(z_n)| d_{(1+\epsilon)}(z_n) < \infty$. (b) $\sup_{z \in E_{(1+\epsilon),\varphi_r}} \lim_{n\to\infty} |V(z_n)| d_{(1+\epsilon)}(z_n) < \infty$.

(i) **Proof.** The researcher assumes that $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. In the following we prove (i)-(iii). Suppose that $D_{(1+\epsilon),\varphi_r} = \theta$. Then (iii) (b) holds. By (iii) of Proposition 2, (iii) also holds. Fix $(z+\epsilon) \in \mathbb{R}$, let $(1+\epsilon)(z) = f_\ltimes^2(z)\sigma_\ltimes(z)$, then $(1+\epsilon) \in \beta^\alpha$. Take $\ltimes = \varphi_r(z+\epsilon)$, $\epsilon = 0$. By the estimate on $\|(1+\epsilon)\varphi_r - (1+\epsilon)\psi_r(1+\epsilon)\|_\beta$, the researcher has

$$C \geq \left|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon) + V(z+\epsilon)\sigma(z+\epsilon)\frac{(1-|\varphi_r(z+\epsilon)|)^\epsilon}{\left(1-\overline{\varphi_r(z+\epsilon)}\,\psi_r(z+\epsilon)\right)^{2\epsilon}} - (1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)\tau^{1+\epsilon}(z+\epsilon) - 2(\epsilon)\overline{\varphi_r(z+\epsilon)}\,(1+\epsilon)\psi_r(z+\epsilon)\tau^\epsilon(z+\epsilon)\sigma(z+\epsilon)\frac{1-|\psi_r(z+\epsilon)|^2}{1-\overline{\varphi_r(z+\epsilon)}\,\psi_r(z+\epsilon)}\right|.$$

Hence

$$\left|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon) - (1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)\tau^{1+\epsilon}(z+\epsilon) - 2(\epsilon)\overline{\varphi_r(z+\epsilon)}\,(1+\epsilon)\psi_r(z+\epsilon)\tau^\epsilon(z+\epsilon)\sigma(z+\epsilon)\frac{1-|\psi_r(z+\epsilon)|^2}{1-\overline{\varphi_r(z+\epsilon)}\,\psi_r(z+\epsilon)}\right|$$
$$\leq C + |V(z+\epsilon)|(1+\epsilon)(z+\epsilon)\left|\frac{(1-|\varphi_r(z+\epsilon)|)^\epsilon}{\left(1-\overline{\varphi_r(z+\epsilon)}\,\psi_r(z+\epsilon)\right)^{2\epsilon}}\right|. \quad (1)$$

With condition (A) and the assume of $|(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)| < \infty$, the researcher has

$$|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon) - (1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)\tau^\epsilon(z+\epsilon)| \leq C. \quad (2)$$

From Remark 1, the researcher obtains

$$\begin{aligned}&|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\\ &\le C+|(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)(1-\tau^{\epsilon}(z+\epsilon)|\\ &\le C, \qquad (3)\end{aligned}$$

therefore

$$\begin{aligned}&|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)|\\ &\le|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\\ &+|(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\le C.\end{aligned}$$

Thus the researcher gets $G_{1+\epsilon,\,\varphi_r}=\theta$ which implies that $D_{1+\epsilon,\psi_r}\subset D_{1+\epsilon,\varphi_r}$. Similarly, the researcher can obtains that $D_{1+\epsilon,\psi_r}\subset D_{1+\epsilon,\varphi_r}$.

Next suppose that $D_{1+\epsilon,\psi_r}\neq\theta$. Multiply inequality (1) by (1+ϵ)$(z+\epsilon)$ and combine with (i) of proposition 2 , the researcher has that

$$|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)\tau^{\epsilon}\,(z+\epsilon)|(1+\epsilon)(z+\epsilon)\le C.$$

Therefore, with Remark 1, the researcher obtains

$$\begin{aligned}&|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\\ &\le|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)\tau^{\epsilon}\,(z+\epsilon)(1+\epsilon)(z+\\ &\epsilon)|+\Big||(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\big(1-\tau^{\epsilon}(z+\epsilon)\big)|\Big|(1+\epsilon)(z+\epsilon)\lesssim C.\end{aligned}$$

Combining with (iii) of Proposition (2), the researcher has that

$$\sup_{z\in\mathbb{R}}|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)|(1+\epsilon)(z+\epsilon)<\infty,\quad \sup_{z\in\mathbb{R}}|(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|<\infty.$$

Hence from inequality (1) again, the researcher has that

$$|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)\tau^{\epsilon}\,(z+\epsilon)|(1+\epsilon)(z+\epsilon)\le C.$$

Applying Remark 1 again, the researcher has

$$|(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)-(1+\epsilon)(z+\epsilon)\psi_r(z+\epsilon)|\leq C.$$

Hence the researcher has that $(1+\epsilon)(z+\epsilon)\varphi_r(z+\epsilon)\to\infty$ which implies that $D_{1+\epsilon,\psi_r}\subset D_{1+\epsilon,\varphi_r}$. Similarly, the researcher can proves that $D_{1+\epsilon,\psi_r}\subset D_{1+\epsilon,\varphi_r}$. Hence the researcher
gets the condition (i)-(iii).

In the following the researcher can proves (iv)-(vi). For $f^2\in\beta^{1+\epsilon}$, by the triangle, the researcher has that

$$\begin{aligned}\left\|((1+\epsilon)C_{\varphi_r}-(1+\epsilon)C_{\psi_r}(f^2)\right\|_{\beta^{1+\epsilon}}&\\ \geq |U(z)f^2(\varphi_r)(z)-V(z)f^2(\psi_r)(z)|&\\ -\|f^2\|_{\beta^{1+\epsilon}}|(1+\epsilon)(z)\varphi_r(z)-(1+\epsilon)(z)\psi_r(z)|.&\end{aligned}$$

Hence

$|U(z)f^2(\varphi_r)(z)-V(z)f^2(\psi_r)(z)|\leq\left\|((1+\epsilon)C_{\varphi_r}-(1+\epsilon)C_{\psi_r}(f^2)\right\|_{\beta^{1+\epsilon}}+$
$C\|f^2\|_{\beta^{1+\epsilon}}$. (4)

Consider the function $g_{\ltimes}$ defined by $g_{\ltimes}(z)=\frac{1}{(1-\bar{\ltimes}z)^{\epsilon}}$ and take $\ltimes=\varphi_r(z+\epsilon),\ \epsilon=0,$ then from the up inequality the researcher has that

$$\left|\frac{U(z+\epsilon)}{(1-(|\varphi_r(z+\epsilon)|)^2)^{\epsilon}}-\frac{V(z+\epsilon)}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}\right|\leq C. \tag{5}$$

Consider the function $f_{\ltimes}^2=\frac{\left(1-|\ltimes|^2\right)^{\epsilon}}{(1-\bar{\ltimes}z)^{2\epsilon}}$ and take $\ltimes=\varphi_r(z+\epsilon),\ \epsilon=0,$ similarly the researcher has that

$$\left|\frac{U(z+\epsilon)}{\left(1-\left(\varphi_r(z+\epsilon)\right)^2\right)^{\epsilon}}-V(z+\epsilon)\frac{(1-(|\varphi_r(z+\epsilon)|)^2)^{\epsilon}}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}\right|\leq C. \tag{6}$$

By (5) which multiplied with $\left|\frac{1-(|\varphi_r(z+\epsilon)|)^2}{1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)}\right|^{\epsilon}$ and (6) , the researcher has that

$$|U(z+\epsilon)|\left|\frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}-\frac{1}{\left(1-\left(\varphi_r(z+\epsilon)\right)^2\right)^{\epsilon}}\right|\leq C. \tag{7}$$

similarly, the researcher has that

$$|V(z+\epsilon)|\left|\frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}-\frac{1}{\left(1-\left(\varphi_r(z+\epsilon)\right)^2\right)^{\epsilon}}\right| \leq C. \tag{8}$$

From inequalities (5) and (7), the researcher has that

$$|U(z)-V(z)|\left|\frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}\right|\leq C. \tag{9}$$

First, suppose that $E_{(1+\epsilon),\varphi_r}=\emptyset$. Then by (9) and Lemma 2. the researcher obtains that

$$\left|V(z+\epsilon)\frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z+\epsilon)\right)^{\epsilon}}\right|<\infty.$$

Apply Lemma 2 again, the researcher has that

$$\sup_{z\in\mathbb{R}}\left|\frac{1}{\left(1-\left(\psi_r(z+\epsilon)\right)^2\right)^{\epsilon}}\right|<\infty.$$

Which implies that $E_{(1+\epsilon),\psi_r}=\emptyset$. and let $\{z_{(1+\epsilon)}\}\in E_{(1+\epsilon),\varphi_r}$. By (5), the researcher gets

$$\left|V(z_{(1+\epsilon)})\right|\left|\frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r\left(z_{(1+\epsilon)}\right)\right)^{\epsilon}}\right|\longrightarrow\infty.$$

Then from (8), the researcher gets that

$$\left|V(z_{(1+\epsilon)})\right|\left|\frac{1}{\left(1-\left(\left|\psi_r(z_{(1+\epsilon)})\right|\right)^2\right)^{\epsilon}}\right|\longrightarrow\infty.$$

This mean that $E_{1+\epsilon,\varphi_r} \subset E_{1+\epsilon,\psi_r}$.Similarly the researcher obtains $E_{1+\epsilon,\psi_r} \subset E_{1+\epsilon,\varphi_r}$. Then the researcher obtains the condition (iv).

Combing (5) and (6), for $\{z_{(1+\epsilon)}\} \in E_{(1+\epsilon),\varphi_r}$,

$$\lim_{\epsilon\to\infty} |V(z_{(1+\epsilon)})| \left| \frac{1}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z_{(1+\epsilon)})\right)^{\epsilon}} \right| \left| 1 - \frac{\left(1-|\varphi_r(z_{(1+\epsilon)})|^2\right)^{\epsilon}}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z_{(1+\epsilon)})\right)^{\epsilon}} \right| \leq C.$$

Thus the researcher gets

$$\left| \frac{\left(1-|\varphi_r(z_{(1+\epsilon)})|^2\right)^{\epsilon}}{\left(1-\overline{\varphi_r(z+\epsilon)}\psi_r(z_{(1+\epsilon)})\right)^{\epsilon}} \right| \to 1. \tag{10}$$

By (7) and (10), for all $\{z_{(1+\epsilon)}\} \in E_{(1+\epsilon),\varphi_r}$, the researcher has that

$$\lim_{\epsilon\longrightarrow\infty} |U(z_{1+\epsilon}) - V(z_{1+\epsilon})| \frac{1}{\left(1-|\varphi_r(z_{(1+\epsilon)})|^2\right)^{\epsilon}} \leq C.$$

Similarly, the researcher obtains

$$\lim_{\epsilon\longrightarrow\infty} |U(z_{1+\epsilon}) - V(z_{1+\epsilon})| \frac{1}{\left(1-|\psi_r(z_{(1+\epsilon)})|^2\right)^{\epsilon}} \leq C. \tag{11}$$

Thus the researcher gets the condition (v).

From (4), the researcher obtains that there exists a positive constant C such that for any $\{z_{(1+\epsilon)}\} \in E_{(1+\epsilon),\varphi_r}$, and any $f^2 \in \beta^{1+\epsilon}$ with $|f^2|_{\beta^{1+\epsilon}} \leq 1$,

$$\lim_{\epsilon\to\infty} |U(z_{1+\epsilon})||f^2(\varphi_r)(z_{1+\epsilon}) - f^2(\psi_r)(z_{1+\epsilon})| \leq C.$$

Therefore, the researcher gets

$$\sup_{\{z_{(1+\epsilon)}\} \in E_{(1+\epsilon),\varphi_r}} \lim_{\epsilon\to\infty} |U(z_{1+\epsilon})|_{d_{1+\epsilon}}(z_{1+\epsilon}) \leq C.$$

Hence condition (vi) holds.

Conversely, the researcher supposes that the condition (i)-(vi) hold. Let $\partial E_{1+\epsilon}, \varphi_r$ be the set of the cluster points of each $\{z_{(1+\epsilon)}\} \in E_{(1+\epsilon),\varphi_r}$. Then $\partial E_{1+\epsilon}, \varphi_r \subset \partial\mathbb{R}$. There exists a constant $M_1 > 0$ and a compact subset N of $\overline{\mathbb{R}}$ such $\partial E_{1+\epsilon}, \varphi_r \subset N$,

$$\sup_{z \in N} |U(z) - V(z)| \frac{1}{\left(1 - \left|\varphi_r\left(z_{(1+\epsilon)}\right)\right|^2\right)^{\epsilon}} + \sup_{z \in N} |V(z)|_{d_{1+\epsilon}}(z) \leq M_1,$$

and

$$\sup_{z \in N} |U(z) - V(z)| \frac{1}{\left(1 - \left|\psi_r\left(z_{(1+\epsilon)}\right)\right|^2\right)^{\epsilon}} + \sup_{z \in N} |U(z)|_{d_{1+\epsilon}}(z) \leq M_1.$$

On the other hand, there exists a constant $M_2 > 0$ such that

$$\sup_{z \in \mathbb{R}/N} |U(z)| \frac{1}{\left(1-\left|\varphi_r\left(z_{(1+\epsilon)}\right)\right|^2\right)^{\epsilon}} + \sup_{z \in \mathbb{R}/N} |V(z)| \frac{1}{\left(1-\left|\psi_r\left(z_{(1+\epsilon)}\right)\right|^2\right)^{\epsilon}} \leq M_2.$$

Then the researcher obtains that for $M_0 = \{M_1, M_2\}$,

$$\sup_{z \in \mathbb{R}} L(z) < M_0.$$

Hence, by proposition 1, $(1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}$ is bounded from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$.

4- Results

1- The weighted operator $(1+\epsilon)C_{\varphi_r}$ is compact from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$ if and only $G_{(1+\epsilon),\varphi_r} = G_{(1+\epsilon),\varphi_r} = \theta$.

2- To characterize the compactness of $(1+\epsilon)C_{\varphi_r} - (1+\epsilon)C_{\psi_r}$ from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$, the researcher needs the assumption

$$\frac{|U(Z_n)|p_n}{(1-|\varphi_{rn}|^2)^{\epsilon}} \longrightarrow 0 \qquad and \qquad \frac{|V(Z_n)|p_n}{(1-|\psi_{rn}|^2)^{\epsilon}} \longrightarrow 0$$

5- Recommendations

1- The researcher supposes that $uC_{\varphi_r} - vC_{\psi_r}$ is compact from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$. Consider the function defined by $f_{m,k}^2(z) = f_{\ltimes}^2(z)\left(\sigma_{\ltimes}^m(z) - \ltimes^{m-k}\sigma_{\ltimes}^k(z)\right)$, for $\in \mathbb{R}$ and nonnegative integers k, m with $m > k \geq 1$.

2- The researcher supposes that $C_{u,\varphi_r} \neq \emptyset$ and let $\{z_n\} \in C_{u,\varphi_r}$. Then

$|u(z_n)\varphi_r(z_n)\sigma_n^2| = |u(z_n)\varphi_r(z_n)|\rho_n^2 \to 0.$ Implies that $\rho_n \to 0$ or

$|p_n^2| = 1 - |\tau_n| \to 1.$

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